Stability Properties of Differentially Rotating Flows
In Astrophysical Disks

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Gravity + Spin = Disks
The Problem: Why do Disks accrete?

- Disks are systems whose energy generation mechanism is the release of gravitational energy as gas spirals in.
- Infall requires angular momentum transport.
  - Angular momentum transport is the central dynamical issue in understanding disk structure and evolution.
Astrophysical Jets

- Young stellar objects
- X-ray binaries – accreting NS or BH
- Symbiotic stars – accreting WD
- Supersoft X-ray sources – accreting WD
- AGN – accreting supermassive BH
- Gamma ray burst systems

The Ubiquity of Jets suggests that they are produced under general conditions.

Gravity + Rotation (disk and/or central star) + Magnetic fields
Transport Mechanisms in Accretion Disks: Possibilities

- External torques:
  - MHD winds
  - Tides (non-local)
- Internal Torques
  - Viscosity: molecular, radiative viscosities too small
  - Turbulence: what produces turbulence?
  - Global waves: what excites the waves? (Non-local)
  - Magnetic fields: when are fields important?

Problem is side-stepped using a dimensional parameterization, stress = $\alpha P$
Investigating the Problem: From Global to Local
Disk Equations in Local Co-rotating Frame

Go to radius $R$ rotating at angular velocity $\Omega$ and use local Cartesian coordinates

$$
\rho \frac{\partial v}{\partial t} + (\rho v \cdot \nabla)v = -\nabla P - 2\Omega \times v + 2q\Omega^2 x \hat{x}
$$

Angular velocity distribution, $q=1.5$ for Keplerian, $q=2$ for constant angular momentum

$$
q \equiv -\frac{d \ln \Omega}{d \ln R}
$$

Direct numerical simulations: finite difference, flux-conservative, spectral....
Why Hydrodynamic Turbulence does not work for disks
Hydrodynamic equations for fluctuation velocities

\[ \rho \left( \frac{Du_R}{Dt} - 2\Omega u_\phi \right) = -\frac{\partial P}{\partial R} + \eta V \nabla^2 u_R \]

\[ \rho \frac{Du_z}{Dt} = -\frac{\partial P}{\partial z} - \rho \frac{\partial \Phi}{\partial z} + \eta V \nabla^2 u_z \]

\[ \rho \left( \frac{Du_\phi}{Dt} + \frac{\kappa^2}{2\Omega} u_R \right) = -\frac{1}{R} \frac{\partial P}{\partial \phi} + \eta V \nabla^2 u_\phi \]

\[ \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla + \Omega \frac{\partial}{\partial \phi} \]

\[ \kappa^2 \equiv \frac{1}{R^3} \frac{d(R^4 \Omega^2)}{dR} \]  

Epicyclic Frequency
Hydrodynamic Disk Fluctuations

\[ \frac{1}{2} \frac{\partial}{\partial t} \langle \rho u_R^2 \rangle = 2 \Omega \langle \rho u_R u_\phi \rangle - \langle u_R \frac{\partial P}{\partial R} \rangle - \text{LOSSES} \]

\[ \frac{1}{2} \frac{\partial}{\partial t} \langle \rho u_\phi^2 \rangle = -\Omega(2 + \frac{d \ln \Omega}{d \ln R}) \langle \rho u_R u_\phi \rangle - \langle \frac{u_\phi \partial P}{R \partial \phi} \rangle - \text{LOSSES} \]

\[ \frac{1}{2} \frac{\partial}{\partial t} \langle \rho u^2 \rangle + \langle \nabla \cdot Pu \rangle = -\frac{d \Omega}{d \ln R} \langle \rho u_R u_\phi \rangle - \text{LOSSES} \]

A positive Reynolds stress is not compatible with sustaining the angular fluctuation velocities themselves.
Constant $\ell$ Disk and a Shear Layer

• For a long time the nonlinear instability seen in shear layers was assumed to carry over to disks
• Constant angular momentum disk has $2\Omega = -d\Omega/d\ln R$ – there are no epicyclic oscillations, $\kappa = 0$.
• Modes that are destabilized in shear layers are neutral modes; the sum of linear-amplitude forces is zero
• In generic disks, epicyclic motion is highly stabilizing; the nonlinear behavior of shear layers is special
• Numerical simulations easily find nonlinear instabilities in shear layers and constant angular momentum profiles. No such instabilities are seen in the equivalent disk simulations
Evolution of the “Papaloizou-Pringle” Instability
Magnetized Fluid
Magnetized Disk

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \rho u_\phi^2 + \rho u_{A\phi}^2 \right) = \\
-2\Omega \langle \rho u_R u_\phi \rangle - \frac{d \ln \Omega}{d \ln R} \langle \rho \left( u_R u_\phi - u_{AR} U A\phi \right) \rangle \\
- \left\langle \frac{u_\phi}{R} \frac{\partial}{\partial \phi} \left( P + B^2 / 8\pi \right) \right\rangle - \text{LOSSES}
\]

\[
\frac{1}{2} \frac{\partial}{\partial t} \left( \rho u^2 + \rho u_A^2 \right) = \\
- \frac{d \Omega}{d \ln R} \left\langle \rho \left( u_R u_\phi - u_{AR} u_A \phi \right) \right\rangle + \langle \nabla \cdot P u \rangle - \text{LOSSES}
\]
Orbital Dynamics
Higher angular momentum = lower angular velocity
Orbital Dynamics
Higher angular momentum = lower angular velocity

Spring transfers angular momentum from lower spacecraft to higher
Magnetorotational Instability

- Stability requirement is

\[(k \cdot v_A)^2 > -\frac{d\Omega^2}{d\ln R}\]

- One can always find a small enough wavenumber \(k\) so there will be an instability unless

\[\frac{d\Omega^2}{d\ln R} > 0\]
MRI maximum growth

- Maximum unstable growth rate:
  \[ |\omega_{max}| = \frac{1}{2} \left| \frac{d\Omega}{d \ln R} \right| \]

- Maximum rate occurs for wavenumbers

- For Keplerian profiles maximum growth rate and wavelengths:
  \[ (k \cdot v_A)_{max}^2 = - \left( \frac{1}{4} + \frac{\kappa^2}{16\Omega^2} \right) \frac{d\Omega^2}{d \ln R} \]
  \[ |\omega_{max}| = \frac{3}{4} \Omega \]
  \[ (k \cdot v_A)_{max} = \frac{\sqrt{15}}{4} \Omega \]

Most unstable wavelength is ~ the distance an Alfven wave travels in an orbit.
Maximum e-folding time ~ \( \Omega^{-1} \)
MRI Growth Rate – vertical field case

Keplerian shear, various $k_z$, $k_r$ values

Max growth $0.75 \Omega$

Growth rate $\sim k v_A$

Weak fields

Strong fields
Toroidal Field MRI

- Toroidal fields are unstable to nonaxisymmetric modes
- In a shearing fluid the radial wave vector $k_r$ is time dependent
- Mode growth occurs for small values of $k/k_z$
- Instability favors large $k_z$ values
- Growth time is limited but growth can be huge
- Stability limit when $v_A \sim$ orbital velocity – huge magnetic field
Summary: The MRI

The MRI is important in accretion disks because they are locally hydrodynamically stable by the Rayleigh criterion, $dL/dR > 0$, but are MHD unstable when $d\Omega^2/dR < 0$

The MHD instability is:

- Local
- Linear
- Independent of field strength and orientation

The measure of the importance of a magnetic field is not determined solely by the value of $\beta = P_{\text{gas}} / P_{\text{mag}}$

Magnetic fields do not go away when the MRI is stabilized by a strong field!
Magnetic Fields Alter the Fundamental Characteristic of a Fluid: Magnetic Hoiland Criteria

Adiabatic, unmagnetized:

\[ N^2 + \kappa^2 > 0 \]

\[- \left( \frac{\partial P}{\partial Z} \right) \left( \frac{\partial l^2}{\partial R} \frac{\partial \ln P \rho^{-5/3}}{\partial Z} - \frac{\partial l^2}{\partial Z} \frac{\partial \ln P \rho^{-5/3}}{\partial R} \right) > 0 \]

Adiabatic, magnetized (Balbus 1995):

\[ N^2 + \frac{\partial \Omega^2}{\partial \ln R} > 0 \]

\[- \left( \frac{\partial P}{\partial Z} \right) \left( \frac{\partial \Omega^2}{\partial R} \frac{\partial \ln P \rho^{-5/3}}{\partial Z} - \frac{\partial \Omega^2}{\partial Z} \frac{\partial \ln P \rho^{-5/3}}{\partial R} \right) > 0 \]

\( N^2 \) is the Brunt-Vaisala frequency

\( \Omega \) replaces \( \ell \)!
“Departures from uniform rotation and isothermality are indeed a source of dynamic instability. It is just that magnetic tension and magnetically confined conduction are needed to provide the right coupling to tap into these sources.” – Balbus 2001
When does the MRI act like an $\alpha$ model?

- The $\alpha$ model assumes the disk is thin: energy released is radiated locally.
- The $\alpha$ model assumes that the same stress transporting angular momentum produces the heat – rapid thermalization.
- Turbulence is, however, not a viscosity – Reynolds and Maxwell stress have different properties from viscosity.
- Stress is not set by $\alpha$ P.

\[
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\]
\[
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\]
\[
- \left\langle \frac{u_\phi}{R} \frac{\partial}{\partial \phi} \left( P + \frac{B^2}{8\pi} \right) \right\rangle - \text{LOSSES}
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\]

Balbus & Papaloizou 1999
Shearing box dynamo – Space-time diagrams

Radial field

Toroidal field

Simon et al 2010
Non-ideal fluid applications
Non-ideal MRI

- Resistivity: field slips through fluid – balance where $k \nu_A \sim k^2 \eta$
- Viscosity: viscous damping prevents fluid motion – balance where $\Omega \sim k^2 \nu$
- Flow characterized by Reynolds number, $Re = \frac{v_A^2}{\nu \Omega}$, Magnetic Reynolds number, or Elsasser number $Re_m = \frac{v_A^2}{\eta \Omega}$
- Nominally, MRI suppressed for $Re = 1$, $Re_m = 1$; simulations show significant effects for much larger Reynolds numbers
Stress increases with increasing $Pr_m$

- The flow is characterized by the magnetic Prandtl number, $Pr = v/\eta$
- Turbulent saturation level affected by value of $Pr$ – does this carry over into Nature even in highly ionized disks? (e.g., Potter & Balbus 2014; Balbus & Lesaffre 2008; Balbus & Henri 2008)
MRI Summary

- Does the MRI lead to disk turbulence?
  - Yes

- At what level does the turbulence saturate?
  - Subthermal levels, but otherwise?

- What are the characteristics of the turbulence?
  - Anisotropic, correlated fluctuations; radial angular momentum transport is the cause of the MRI!

- How, when and where does the energy thermalize?
  - Eddy turnover timescale $\sim \Omega^{-1}$

- Are large-scale magnetic fields created?
  - Local (and global) disk simulations show an $\alpha$–$\Omega$ dynamo effect

- How does the MRI behave for non-ideal plasmas?
  - Generally similarly, but with qualitative and quantitative differences

- Is net flux transported through the disk?
  - Not by the turbulence apparently, but possibly through global motions