Modeling scalar transport in vorticity dominated flows

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“Turbulence is the graveyard of theories.”
— H. W. Liepmann
Simplest turbulence problem:

What are the characteristics of small-scale turbulent motions, how do these depend on the properties of the large-scale motions from which they derive, and knowing them, how can we model the transport of scalar and vector quantities, such as concentration, energy, or momentum?

Formulate a statistical description of transport in the presence of coherent vortical structures:

Understand the statistics of turbulent flow transport in physical space (accounting for phase relationships).
The simplest first step in turbulence modeling,

**A scalar transport model**

**GOAL:** Determine the expectation value of the scalar concentration and its variance (perhaps higher moments) as a function of time and space, *knowing only the statistical properties of the flow.*

Transport of a scalar quantity, $c$.

\[
\frac{\partial c}{\partial t} + u \cdot \nabla c = S(x,t)
\]

\[
c(x,t) = \int S(x',t') \ G(x,t | x',t') \ dx'dt'
\]

\[
\langle c(x,t) \rangle = \int S(x',t') \ P(x,t | x',t') \ dx'dt'
\]

\[
P(x,t | x',t') \text{ is the probabilistic Green function}
\]

\[
\langle c^2(x,t) \rangle = \int S(x_1,x_2,t_1,t_2)P(x,x,t,t | x_1,x_2,t_1,t_2)dx_1dx_2dt_1dt_2
\]

\[
P(x,x,t,t | x_1,x_2,t_1,t_2) \text{ is the time reversed generalized pair dispersion}
\]
Lagrangian statistics in turbulent flows:

- Velocity measurements follow an approximately Gaussian distribution
- For small temporal increments velocity-difference PDFs are highly non-Gaussian

Laboratory water flow with Reynolds number of up to 63000
Acceleration up to 1500g measured
A Two-Dimensional Analog of 3D Turbulence:

Point (line) vortex:

\[ u_\theta \sim \frac{1}{r} \]

\[ u_r = u_z = 0 \]

\[ u(x) = \sum_{k=1}^{N} \frac{1}{2} \frac{\hat{z} \cdot (x - x_k)}{|x - x_k|^2} \]

Mininni et al. (2008+)

Gruchalla et al. (2009)

Rast & Pinton (2009)
Point vortex simulations:

\[ u(x) = \frac{1}{2} \sum_{k=1}^{N} \frac{k}{|x - x_k|} \left( \wedge \left( \begin{bmatrix} x \\ x_k \end{bmatrix} \right) \right) \]

Merger of close vortices

\[ u_{\theta} \sim \frac{1}{r} \]

Stirring by vortex creation

\[ u_r \sim 0 \]

“Trapping” events at all scales dominate

Lagrangian single point and pair dispersion statistics

Rast & Pinton 2009
- “circulation conserving” mergers
- strong inverse cascade
- secular increase of the interaction energy

- “energy conserving” mergers
- weak, if any, inverse cascade
- fluctuations in interaction energy due to sign imbalance in vortex creation events

- “energy conserving” mergers
- vortex creation circulation distribution has three times the fwhm
- weak, if any, inverse cascade
- fluctuations in interaction energy due to sign imbalance in vortex creation events
\[ \langle c(x,t) \rangle = S(x',t') \, P(x,t | x',t') \, dx'dt' \]

Measure \( P(x,t | x',t') \) in point vortex simulations:

If one had this, the problem would be solved exactly.

Instead take: \( P(x,t | x',t') = \frac{P(r,t)}{2 \, r} \) (isotropy)

\( P(r,t) \) is the probability of traveling an Eulerian distance \( r \) in time \( t \) along a Lagrangian path.

REVISED GOAL: Model \( P(r,t) \) knowing only the statistical properties of the flow?
What is to be achieved?

Statistics of turbulent flow
(point vortex analog)

\[
Prob(x, t \mid x', t') = \frac{Prob(r, t)}{2r}
\]  
(homogeneity)

\[
\langle c(x, t) \rangle = S(x', t') P(x, t \mid x', t') \, dx' \, dt'
\]

For any source distribution.
$P(r,t)$:

- Probability distribution of the Eulerian distance traveled along a Lagrangian trajectory approximates that of a random walk in the plane with variable step size.

- Variance $s^2 \sim r^2 \log(P(r,t))$ is a function of time.

$P(r,t) \sim re^{-r^2/\sigma(t)^2}$
The average value of $r$ scales ballistically and then diffusively with time:

\[
\langle r^2 \rangle \sim \sigma^2 \sim t \quad \text{for } t > T_L
\]
(diffusive regime)

\[
\langle r^2 \rangle \sim \sigma^2 \sim t^2 \quad \text{for } t < T_L
\]
(ballistic regime)

Lagrangian autocorrelation time

Eulerian autocorrelation time
Turbulence is not diffusive, in the broader sense (non-random walk):

Motions in inertial range on average undershoot a random walk

\( t > T_L \): motions on scales larger than the integral scale on average overshoot a random walk

\( t < \tau_K \): motions on the Kolmogorov scale on average overshoot a random walk
Can we make a statistical model of the point vortex transport pdfs?

- Can we model the deviations from random walk distributions using only measurable statistics of the flows?
- Does anything from such a model translate over to three-d turbulence?
What do Lagrangian tracers in simulations of three–dimensional homogeneous isotropic turbulence show?

Mininni et al. (2008+)

\[ \left\langle r^2 \right\rangle \sim t^2 \] (ballistic regime)

\[ \left\langle r^2 \right\rangle \sim t \] (diffusive regime)

average value of \( r \) scales ballistically and then diffusively with time
Point vortex model:

Three-dimensional DNS:
What can spectra capture?

\[
\begin{align*}
\hat{A} &= \tilde{u}_x \tilde{u}_x^* = \tilde{u}_y \tilde{u}_y^* \\
\hat{C} &= \tilde{u}_x \tilde{u}_y^* \\
\tilde{u}_x &= \sqrt{\hat{A}} \exp(i x) \quad \text{Random phase yields Gaussian distributed values} \\
\tilde{u}_y &= \tilde{u}_x \exp(i y) \quad \text{Satisfies autocorrelation} \\
\hat{C}_x \hat{C}_y &= \hat{C} \\
\tilde{u}_x \tilde{u}_x^* \exp(i y) &= \hat{C} \\
y &= i \ln \left( \frac{\hat{C}}{\hat{A}} \right)
\end{align*}
\]

Generate many time series that have same auto and cross correlations as Lagrangian trajectories in point vortex flow.
Auto and cross correlations in velocities sufficient to capture deviations in pdf from random walk only at very short times.

All velocity phase information is lost to auto and cross correlation.

"Trapping" events at all scales dominate Lagrangian single point and pair dispersion statistics

Trapping requires information about phase relationship between $u_x$ and $u_y$
Underlying dynamics is circular motion around point vortices or collections of them. (a constraint on the velocity phase relationships)

Incorporate that constraint directly into a random walk.

\[ r = 2r_t \left| \sin \left( \frac{u}{r_t} t \right) \right| \]
Statistical model of the point vortex transport:

Choose as independent random variables $u, r_t,$ and $t_t$ in a random walk of stepsize:

$$r_{\text{step}} = 2r_t \left| \sin \left( \frac{u}{r_t} t_t \right) \right|$$

1. power law distribution of trap radii (Kolmogorov) up to integral length (maximum trapping radius)

$$P(r_t) \propto r_t^{4/3}$$

2. uniform distribution of trapping durations up to integral time (Eulerian autocorrelation time)

$$P(t_t) \propto U(0 \rightarrow t_E)$$

3. Maxwellian (or observed)

Lagrangian velocity distribution

$$P(u) \propto u \exp(-u^2)$$
4. lowest wav number largescale flow (as Eulerian background)
What achieved?

- Simple scalar transport model based on random walk with phase preserving eddies determining the steps.
- Only one eddy component and a large scale structured “mean” flow are needed to describe scalar transport in the point vortex analog.

\[
\langle c(x,t) \rangle = S(x',t') P(x,t \mid x',t') \, dx'dt'
\]

For any source distribution.
Decompose motion into two components:

- Vortical motion around vortex filaments
- Motion along filaments

Construct transport PDFs

- Vortical contribution described with constrained motions, as in 2D
- Tangential component as random walk in 3D (filaments randomly oriented)

Use weighted sum of PDFs based on fraction of time Lagrangian parcel spends in each type of motion

Three-dimensional homogeneous isotropic turbulence shows similar transport behavior.

How might a similar model work in that setting?

Mininni et al. (2008+)
Have successfully built a turbulent scalar transport model:

• Accounts for velocity phase relationships inherently introduced by turbulent coherent structures
• Has two components – eddy constrained random walk and a large scale flow
• Needs four properties of the flow to construct:
  1. maximum trapping radius (for range of Kolmogorov distribution of eddy traps)
  2. Eulerian autocorrelation time (for uniform distribution of trapping times)
  3. Lagrangian velocity distribution and correlation time for high speed tail (if very short time is relevant)
  4. mean flow amplitude and structure